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Beauty and the twist: the Bethe ansatz for twisted $\mathcal{N} = 4$ SYM

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Abstract: It was recently shown that the string theory duals of certain deformations of the $\mathcal{N} = 4$ gauge theory can be obtained by a combination of T-duality transformations and coordinate shifts. Here we work out the corresponding procedure of twisting the dual integrable spin chain and its Bethe ansatz. We derive the Bethe equations for the complete twisted $\mathcal{N} = 4$ gauge theory at one and higher loops. These have a natural generalization which we identify as twists involving the Cartan generators of the conformal algebra. The underlying model appears to be a form of noncommutative deformation of $\mathcal{N} = 4$ SYM.

Keywords: AdS-CFT and dS-CFT Correspondence, Bethe Ansatz.

Contents

1. Introduction

Over the years, the relation between gauge theories and string theory has been an abundant source of understanding of both theories. In the form of the AdS/CFT correspondence [1–3] it provides us with an explicit framework in which such information can be extracted. One of the intriguing and fascinating developments is the integrability of both the gauge theory dilatation operator [4–7] and of the world sheet sigma model [8–10]. Nevertheless, understanding the duality beyond the near-BPS limit remains challenging, even within the set of observables for which integrability plays a dominant role.

The study of the deformations of $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) provides new controlled instances in which the duality may be tested. Given the reduced amount of symmetry, such a setup represents a different way of departing from the near-BPS regime. A natural place to start is provided by the exactly marginal deformations of the theory [11].

Some steps in this direction have been taken in [12, 13] where the Leigh-Strassler or β-deformation has been studied from the standpoint of the one-loop dilatation operator of certain sectors of the theory.

More recently, in [14] it was argued that a sequence of T-duality transformations and coordinate shifts yields the supergravity duals to deformations of $\mathcal{N} = 4$ SYM which preserve all the Cartan generators of the superconformal algebra. This allowed the explicit construction of the supergravity background dual to the β-deformed theory. It turns out that bosonic world sheet theory in this background exhibits a Lax pair [15] similar to the undeformed case [16, 17]. Furthermore, the string Bethe equations derived from its restriction to two world sheet fields agree with the thermodynamic limit of the Bethe equations in the two-spin sector of the deformed theory [18].

The algorithm of [14] was used in [15] for the construction of the supergravity background dual to a three-parameter family of (generically) non-supersymmetric deformations of $\mathcal{N} = 4$ SYM. The reasons implying the integrability of the bosonic world sheet theory in the background dual to β-deformed SYM go through in this more general case as well. In fact, given that the transformations leading to this background are well-defined in string theory, we expect that the proof [19] that the world sheet nonlocal charges survive quantization goes through without major alterations.

It is then of great interest to analyze the field theory duals of general deformations [14, 15], which we will generically refer to as "twisted $\mathcal{N} = 4$ SYM". The gauge theory duals to these backgrounds involve certain phase deformations of the terms appearing in the lagrangian. It turns out that there are in fact more possible deformations than those appearing in this string theory construction and it is interesting to identify those preserving the integrability of the dilatation operator. From a different perspective, the spin chain describing the dilatation operator of $\mathcal{N} = 4$ SYM exhibits many integrable deformations similar in spirit with those described by string theory, yet different; it is interesting to learn which of them can be realized within the class of twisted $\mathcal{N} = 4$ theories.

In this paper we address these issues. To construct the Bethe equations for the complete twisted $\mathcal{N} = 4$ SYM we draw information from the sectors whose hamiltonians can be easily constructed from standard Feynman diagram calculations. In section 2 we start from the three-phase deformation introduced in [15] and show that the phases appearing while reordering fields are determined by the phases appearing in the reordering of fermions. Using this observation we construct an operator whose eigenvalues are these phases, for any representation of the reordered fields. Since this operator twists the spin chain hamiltonian describing the dilatation operator, it can be used to twist the R-matrix of the chain as well. In section 3 we discuss this in detail and show that this operator generates an isomorphism of the space of solutions of the Yang-Baxter equations. For the sectors of $\mathcal{N} = 4$ SYM described by spin chains in the fundamental representation of a unitary group such twisted R-matrices reduce to those discussed in [12]. The largest sector with this property is the $\mathfrak{su}(2|3)$ sector of the undeformed theory, see [20] for a detailed account. In section 4 we review the derivation of the nested Bethe Ansatz for the $\mathfrak{su}(2|3)$ sector of $\mathcal{N} = 4$ SYM and contrast it with the nested Bethe ansatz for the twisted theory. The main conclusion of this analysis is that the Bethe equations are the same as those obtained by twisting the diagonalized magnon S-matrix with the same twist operator used to twist the R-matrix. We also discuss in principle the steps necessary to apply the nested Bethe ansatz algorithm for other sectors of the theory, not described by spin chains in the fundamental representation. Rather than following this line, in section 5 we make use of the observation that the twist operator can be applied directly to the magnon scattering matrix and discuss general flavor-dependent twists of the $\mathcal{N}=4$ spin chain. We show that the most general such twist which has a lagrangian realization within $\mathcal{N}=4$ SYM is the one constructed in [15]. Using different dual presentations of the Dynkin diagram we identify various choices of the twist parameters preserving supersymmetry and non-abelian global symmetry. The same twist operation leads us to the conjecture that the higher-loop Bethe equations are a twisted form of those conjectured in [21] for the $\mathcal{N}=4$ SYM theory. Interestingly, the consistency of this conjecture relies on the the same details as the compatibility of the twist with the Feynman rules at the one-loop level. Last, we discuss deformations which break Lorentz and conformal invariance. While it is not completely clear what is the structure of the lagrangian of the deformed field theory, it is relatively straightforward to write the Bethe equations for its dilatation operator; we spell them out in section 6. Then, based on the naive application of the construction [14], we discuss the possible structure deformed lagrangian as well as that of the eigenvectors of the dilatation operator. section 7 contains further discussions.

2. Charges and twists

As mentioned in the introduction, we are interested in analyzing a certain class of integrable deformations of the $\mathcal{N}=4$ spin chain and identifying those which correspond to rescaling terms of the lagrangian by nontrivial constant phases. We will draw information from examples involving closed subsectors of the theory, in which it is easy to explicitly construct the dilatation operator starting from the lagrangian. In this context we consider deformations which can be realized as a Moyal-like $*$-product based on the $\mathfrak{su}(4)$ Cartan charges of fields, introduced following [14, 15]. The gravity dual of the most general such deformation was constructed in [15] and was also argued that the corresponding world sheet theory is classically integrable.

The easiest sectors to analyze are those described by spin chains in the fundamental representation of some unitary group, as the dilatation operator acts only by changing the order of neighboring fields in gauge invariant operators, with certain weights. The deformation of the lagrangian implies that these weights acquire nontrivial phases determined by the $\mathfrak{su}(4)$-charges of the fields being reordered. For each pair of fields they can be determined from standard Feynman diagram calculations. However, due to the nature of the deformation and the fact that the $\mathfrak{su}(4)$-charges of all fields are determined by those of the fermionic fields, the phase obtained by reordering any two fields is the same as the phase obtained by reordering monomials constructed out of fermions and carrying the same charges as the initial fields. For example, the scalars Φ_A are in the $\mathbf{6}$ of $\mathfrak{su}(4)$ $\Phi_A \sim \Phi_{ab}$ and thus they have the same $\mathfrak{su}(4)$-charges as the fermion bilinears $\Psi_a\Psi_b$. We will use an

X	$\psi_{1\alpha}$	$\psi_{2\alpha}$	$\psi_{3\alpha}$	χ_α	$A_{\alpha\dot{\alpha}}$	ϕ_1	ϕ_2	ϕ_3
q^1_X	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	1	0	0
q^2_X	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	1	0
q^3_X	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	1

Table 1: Charges of the fundamental fields under the Cartan generators of $\mathfrak{su}(4)$.

$\mathfrak{su}(3)$ ($\mathcal{N}=1$) notation where a complex triplet of scalars ϕ_a is defined as $\phi_a = \Phi_{a4}$ and $\bar{\phi}^a = \frac{1}{2}\varepsilon^{abc}\Phi_{bc}$ and we will denote the triplet of fermions by $\psi_a = \Psi_a$, while $\chi = \Psi_4$ is the gluino.

This simple structure is somewhat modified in the sectors for which the charges of the initial excitations are reorganized by the interactions. Though more complicated, the deformation of the spin chain hamiltonian can still be determined from Feynman diagrams in terms of the R-charges of the fundamental fermions. As we will see in detail later, the Bethe ansatz presents us with a simple way of bypassing this slight complication. Indeed, in the undeformed theory, the operators creating excitations corresponding to the simple roots of the $\mathfrak{psu}(2,2|4)$ in the $\mathcal{N}=4$ spin chain exhibit diagonal scattering and thus no rearrangement of their $\mathfrak{su}(4)$-charges.

To summarize, we need to find the phases obtained by interchanging the position of the fermions. They can be found easily by starting from the deformed $\mathcal{N}=4$ lagrangian, which is obtained by replacing all commutators with deformed commutators

$$[X,Y]_{\mathbf{C}} = e^{i\mathbf{q}_X\times\mathbf{q}_Y/2}\,XY - e^{i\mathbf{q}_Y\times\mathbf{q}_X/2}\,YX \tag{2.1}$$

or, alternatively, the structure constants are replaced with

$$f_{abc}X^bY^c \;\mapsto\; \left(\cos\left(\mathbf{q}_X\times\frac{\mathbf{q}_Y}{2}\right)f_{abc} + i\sin\left(\mathbf{q}_X\times\frac{\mathbf{q}_Y}{2}\right)d_{abc}\right)X^bY^c\,. \tag{2.2}$$

Here we have introduced the antisymmetric $\mathbf{C}$-product

$$\mathbf{q}_X\times\mathbf{q}_Y = \mathbf{q}_X^{\mathrm{T}}\mathbf{C}\mathbf{q}_Y = C_{ab}q^a_Xq^b_Y\,. \tag{2.3}$$

The $\mathfrak{su}(4)$ Cartan charges q^a_X (labeled by $a=1,2,3$) of the fundamental fields are given in table 1. We define the matrix of phases $\mathbf{C}$ as

$$\mathbf{C} = \begin{pmatrix} 0 & -\gamma_3 & +\gamma_2 \\ +\gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & +\gamma_1 & 0 \end{pmatrix}. \tag{2.4}$$

When permuting two fermions Ψ_a and Ψ_b we pick up a phase

$$\Psi_a\Psi_b \mapsto e^{i\mathbf{q}_a\times\mathbf{q}_b}\Psi_b\Psi_a = e^{iB_{ab}}\Psi_b\Psi_a\,, \tag{2.5}$$

where the phase matrix $\mathbf{B}$ in the basis $(\Psi_1,\Psi_2,\Psi_3,\Psi_4)$, i.e. $B_{ab} = \mathbf{q}_a\times\mathbf{q}_b$, is given by

$$\mathbf{B} = \begin{pmatrix} 0 & -\frac{1}{2}(\gamma_1+\gamma_2) & +\frac{1}{2}(\gamma_3+\gamma_1) & +\frac{1}{2}(\gamma_2-\gamma_3) \\ +\frac{1}{2}(\gamma_1+\gamma_2) & 0 & -\frac{1}{2}(\gamma_2+\gamma_3) & +\frac{1}{2}(\gamma_3-\gamma_1) \\ -\frac{1}{2}(\gamma_3+\gamma_1) & +\frac{1}{2}(\gamma_2+\gamma_3) & 0 & +\frac{1}{2}(\gamma_1-\gamma_2) \\ -\frac{1}{2}(\gamma_2-\gamma_3) & -\frac{1}{2}(\gamma_3-\gamma_1) & -\frac{1}{2}(\gamma_1-\gamma_2) & 0 \end{pmatrix}. \tag{2.6}$$

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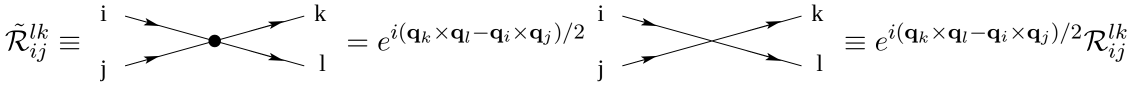

Figure 1: Generic phase deformation of the R-matrix.

These phases can be used to reconstruct the phase matrix for objects with multiple scalar indices like the scalars $\Phi_{ab} \sim \Psi_a\Psi_b$ by summing up the contributions for permuting the individual indices. For example, the phases among scalars $\phi_a = \Phi_{a4} \sim \Psi_a\Psi_4$ are given by the original phase matrix $\mathbf{q}_{a4} \times \mathbf{q}_{b4} = C_{ab}$. Another example is the composite $\Psi_1\Psi_2\Psi_3\Psi_4$ for which all charges vanish. Therefore the matrix $\mathbf{B}$ in (2.6) annihilates the vector $(1, 1, 1, 1)$ as can be easily verified.

3. The deformed R-matrix

Let us consider a $\mathbb{Z}_2$-graded set of states, labeled collectively by i and with the grade of the i-th state denoted by $[i]$. Let us also suppose that there exists a solution $\mathcal{R}$ of the graded Yang-Baxter equation

$$(-)^{[j_2]([j_1]+[k_1])}\mathcal{R}_{i_1i_2}^{j_1j_2}(u-v)\mathcal{R}_{j_1i_3}^{k_1j_3}(u)\mathcal{R}_{j_2j_3}^{k_2k_3}(v) = (-)^{[j_2]([i_1]+[j_1])}\mathcal{R}_{i_2i_3}^{j_2j_3}(v)\mathcal{R}_{i_1j_3}^{j_1k_3}(u)\mathcal{R}_{j_1j_2}^{k_1k_2}(u-v) \tag{3.1}$$

which is labeled by these states. Assuming that the states carry conserved charges denoted by the charge vector $\mathbf{q}$, i.e. $\mathbf{q}_i + \mathbf{q}_j = \mathbf{q}_k + \mathbf{q}_l$, we will show[1] that

$$\tilde{\mathcal{R}}_{ij}^{lk} = e^{i(\mathbf{q}_k\times\mathbf{q}_l-\mathbf{q}_i\times\mathbf{q}_j)/2}\mathcal{R}_{ij}^{lk} \tag{3.2}$$

is also a solution of the graded Yang-Baxter equation. Pictorially, this deformation can be represented as in figure 1.

The number of such conserved charges depends, of course, on the details of the undeformed R-matrix. Usually it may be identified with the rank of the symmetry algebra (plus one for the conserved length of the spin chain). In the simplest cases, for spin chains in the fundamental representation of a unitary group, there are as many such charges as spin orientations. Clearly however, this is not the generic situation and the number of such charges is typically smaller than the dimension of the representation labeling the R-matrix. Nevertheless, the deformation (3.2) leads to a solution of the Yang-Baxter equation.

The Yang-Baxter equation can be checked directly by noticing that, after plugging (3.2) into (3.1), the contribution of the exponential factors is independent on the summation indices. Explicitly, for fixed labels j_1, j_2, j_3 in figure 2, the phases on the two sides of the Yang-Baxter equation are:

$$\begin{aligned}&(\mathbf{q}_{j_2}\times\mathbf{q}_{j_1}-\mathbf{q}_{i_1}\times\mathbf{q}_{i_2})+(\mathbf{q}_{j_3}\times\mathbf{q}_{k_1}-\mathbf{q}_{j_1}\times\mathbf{q}_{i_3})+(\mathbf{q}_{k_3}\times\mathbf{q}_{k_2}-\mathbf{q}_{j_2}\times\mathbf{q}_{j_3}),\\&(\mathbf{q}_{j_3}\times\mathbf{q}_{j_2}-\mathbf{q}_{i_2}\times\mathbf{q}_{i_3})+(\mathbf{q}_{k_3}\times\mathbf{q}_{j_1}-\mathbf{q}_{i_1}\times\mathbf{q}_{j_3})+(\mathbf{q}_{k_2}\times\mathbf{q}_{k_1}-\mathbf{q}_{j_1}\times\mathbf{q}_{j_2}),\end{aligned} \tag{3.3}$$

[1]It turns out that this is a particular case of Reshetikhin's construction of multiparametric quantum algebras [22].

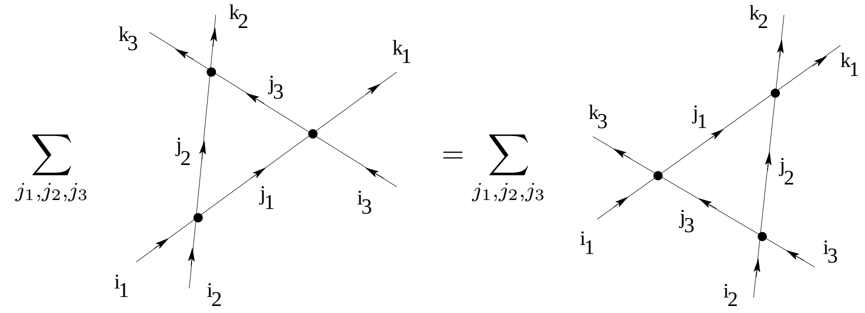

Figure 2: The Yang-Baxter equation.

both of which equal

$$\mathbf{q}_{k_2} \times \mathbf{q}_{k_1} + \mathbf{q}_{k_3} \times \mathbf{q}_{k_1} + \mathbf{q}_{k_3} \times \mathbf{q}_{k_2} - \mathbf{q}_{i_1} \times \mathbf{q}_{i_2} - \mathbf{q}_{i_1} \times \mathbf{q}_{i_3} - \mathbf{q}_{i_3} \times \mathbf{q}_{i_2}\,. \tag{3.4}$$

In showing this equality we only used the fact that the charges labeling the states are conserved and that the $\times$-product is antisymmetric. We see that the combination of the deformations of the individual R-matrices is independent of the intermediate states and equal on the two sides of the Yang-Baxter equation which is therefore satisfied.

More formally, one may notice that the deformation (3.2) is similar to the Moyal-deformation of field theories. From this standpoint the Yang-Baxter equation may be interpreted as the equality of two planar one-loop Feynman diagrams.[2] Then, [23] implies that the noncommutative deformation does not affect planar diagrams. Thus, it follows that the deformation (3.2) affects only the external states in the Yang-Baxter equation.

The hamiltonian of the deformed chain can be constructed following standard rules. Starting from the deformed monodromy matrix

$$\tilde{\mathcal{T}}^{b;\beta_1\ldots\beta_L}_{a;\alpha_1\ldots\alpha_L} = \tilde{\mathcal{R}}^{b_{L-1}\beta_L}_{a\;\;\alpha_L}\tilde{\mathcal{R}}^{b_{L-2}\beta_{L-1}}_{b_{L-1}\alpha_{L-1}}\ldots\tilde{\mathcal{R}}^{b_1\beta_2}_{b_2\alpha_2}\tilde{\mathcal{R}}^{b\,\beta_1}_{b_1\alpha_1}\,\exp\left(i\pi\sum_{i=1}^{L}\sum_{j=1}^{i-1}([\alpha_i]+[\beta_i])[\alpha_j]\right)\,, \tag{3.5}$$

the hamiltonian is given by the logarithmic derivative of the transfer matrix $\tilde{\mathcal{T}}(u) = (-)^{[a]}\tilde{\mathcal{T}}^a_a(u)$

$$\tilde{\mathcal{H}} = -i\,\frac{d}{du}\ln\tilde{\mathcal{T}}(u)\Big|_{u=u_*}\,, \tag{3.6}$$

where u_* is the value of the rapidity at which $\tilde{\mathcal{R}}$ becomes the graded permutation operator. It is not difficult to see that using (3.2) this leads to the following deformation of the hamiltonian

$$\tilde{\mathcal{H}}^{kl}_{ij} = e^{i(\mathbf{q}_k\times\mathbf{q}_l-\mathbf{q}_i\times\mathbf{q}_j)/2}\mathcal{H}^{kl}_{ij}\,. \tag{3.7}$$

In the case in which the states labeling the undeformed hamiltonian form the fundamental representation of some unitary group, it is easy to see the effects of the deformation (3.2); since the undeformed hamiltonian is simply the sum between the identity operator

[2] In fact, if the states labeling the R-matrix transform in the fundamental representation of a unitary group, it is possible to write down a lagrangian in which this is explicitly realized. For example, one may consider a complex bosonic theory with the four-point interactions given by $\mathcal{R}^{kl}_{ij}\phi_k\phi_l\bar{\phi}^i\bar{\phi}^j$.

and the graded permutation operator, it follows that the deformed hamiltonian is obtained by multiplying the graded permutation operator by the same deformation as in (3.2) while leaving the identity operator unchanged.

4. The nested Bethe ansatz in the $\mathfrak{su}(2|3)$ sector

Here we will derive the Bethe equations for a spin chain with $\mathfrak{u}(2|3)$ symmetry and subsequently deform them according to the twist of the R-matrix from section 3. This explicit analysis will suggest a generalization of the twisted Bethe equations to arbitrary superalgebras.

The states of this spin chain correspond to gauge theory operators composed from the fields (with some unspecified permutation),

$$|n_1, \ldots, n_5\rangle = (\phi_1)^{n_1}(\phi_2)^{n_2}(\phi_3)^{n_3}(\chi_1)^{n_4}(\chi_2)^{n_5} + \cdots , \tag{4.1}$$

see [20] for a detailed account of this spin chain in connection with gauge theory. The field ϕ_1 will be considered the vacuum and all the other fields are excitations.

4.1 Undeformed case

To track down the effects of the deformation (3.2) on the Bethe equations let us first review the diagonalization of the transfer matrix of the $\mathfrak{su}(2|3)$ sector of $\mathcal{N} = 4$ SYM which proceeds in the standard way, following the algorithm of the nested Bethe Ansatz.

The spin chain of the $\mathfrak{su}(2|3)$ sector of $\mathcal{N} = 4$ SYM is described by the R-matrix in the fundamental representation of $\mathfrak{u}(2|3)$:[3]

$$\mathcal{R}(u) = \frac{1}{u+i}\left(u\mathcal{I} + i\mathcal{P}\right), \tag{4.2}$$

where $\mathcal{I}$ is the identity and $\mathcal{P}$ is the graded permutation operator

$$\mathcal{I}^{kl}_{ij} = \delta^k_i \delta^l_j, \qquad \mathcal{P}^{kl}_{ij} = (-)^{[k][l]}\delta^l_i \delta^k_j \tag{4.3}$$

and can be expressed in terms of the generators of $\mathfrak{u}(2|3)$ and the Cartan-Killing metric:

$$\mathcal{P} = \sum_{i,j} (-)^{[i]+[j]}\, e^j_i \otimes e^i_j \tag{4.4}$$

Picking a vacuum $|0\rangle$ corresponding to $(\phi^1)^L$, the relevant commutation relations following from the (graded) Yang-Baxter equation are:

$$\begin{aligned}
\mathcal{A}(u)\,\mathcal{B}^{i_1}(v) &= f(v-u)\,\mathcal{B}^{i_1}(v)\,\mathcal{A}(u) - g(v-u)\,\mathcal{B}^{i_1}(u)\,\mathcal{A}(v)\,, && (4.5)\\
\mathcal{D}^{i_1}_{k_1}(u)\,\mathcal{B}^{i_2}(v) &= (-)^{[k_1][j_2]}\, r^{i_1 i_2}_{j_2 j_1}(u-v)\, f(u-v)\,\mathcal{B}^{j_2}(v)\,\mathcal{D}^{j_1}_{k_1}(u) - \\
&\quad - (-)^{[k_1][i_1]}\, g(u-v)\,\mathcal{B}^{i_1}(v)\,\mathcal{D}^{i_2}_{k_1}(u)\,, && (4.6)\\
\mathcal{B}^{i_1}(u)\,\mathcal{B}^{i_2}(v) &= r^{i_1 i_2}_{j_2 j_1}(u-v)\,\mathcal{B}^{j_2}(v)\,\mathcal{B}^{j_1}(u)\,, && (4.7)
\end{aligned}$$

[3] At one loop we may take the $\mathfrak{u}(1)$ factor of $\mathfrak{u}(2|3)$ to represent the length of the spin chain.

where

$$r^{i_1 i_2}_{j_1 j_2} = (-)^{[j_1][j_2]}\mathcal{R}^{i_1 i_2}_{j_2 j_1}\,, \qquad g(u) = \frac{i}{u}\,, f(u) = \frac{u+i}{u} \tag{4.8}$$

and the remaining labels $i, j = 1, \ldots, 4$ enumerate the fields $\phi_2, \phi_3, \chi_1, \chi_2$. Making the ansatz that a state with $n = n_2 + n_3 + n_4 + n_5$ excitations is given by

$$|n_1, \ldots, n_5\rangle = f_{i_1 \ldots i_n}\mathcal{B}^{i_1}(u_{1,1}) \ldots \mathcal{B}^{i_n}(u_{1,n})|0\rangle \tag{4.9}$$

and using (4.5)–(4.7), the diagonalization of transfer matrix constructed out of $\mathcal{R}$ is reduced to the diagonalization of the transfer matrix constructed out of r which also satisfies the Yang-Baxter equation and the (wave) functions $f_{i_1 \ldots i_n}$ are the eigenvectors of this (reduced) transfer matrix.

Repeating these steps four times leads to the Bethe equations:

$$\begin{aligned}
1 &= \left(\frac{u_{1,k} - \frac{i}{2}}{u_{1,k} + \frac{i}{2}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_1} \frac{u_{1,k} - u_{1,j} + i}{u_{1,k} - u_{1,j} - i} \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} - \frac{i}{2}}{u_{1,k} - u_{2,j} + \frac{i}{2}}\,,\\
1 &= \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} - \frac{i}{2}}{u_{2,k} - u_{1,j} + \frac{i}{2}} \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} + i}{u_{2,k} - u_{2,j} - i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} - \frac{i}{2}}{u_{2,k} - u_{3,j} + \frac{i}{2}}\,,\\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} - \frac{i}{2}}{u_{3,k} - u_{2,j} + \frac{i}{2}} \prod_{j=1}^{K_4} \frac{u_{3,k} - u_{4,j} + \frac{i}{2}}{u_{3,k} - u_{4,j} - \frac{i}{2}}\,,\\
1 &= \prod_{j=1}^{K_3} \frac{u_{4,k} - u_{3,j} + \frac{i}{2}}{u_{4,k} - u_{3,j} - \frac{i}{2}} \prod_{\substack{j=1\\ j\neq k}}^{K_4} \frac{u_{4,k} - u_{4,j} - i}{u_{4,k} - u_{4,j} + i}\,.
\end{aligned} \tag{4.10}$$

Furthermore, the cyclicity constraint

$$1 = \prod_{k=1}^{K_1} \frac{u_{1,k} + \frac{i}{2}}{u_{1,k} - \frac{i}{2}} \tag{4.11}$$

ensures that the operators described by the states (4.1) are compatible with taking a color trace. Here we have introduced the numbers of Bethe roots K_j specified by

$$K_j = n_{j+1} + \cdots + n_5\,, \qquad n_j = K_{j-1} - K_j\,. \tag{4.12}$$

Finally, the anomalous dimension of a state reads

$$\delta D = g^2 \sum_{k=1}^{K_1} \left(\frac{i}{u_{1,k} + \frac{i}{2}} - \frac{i}{u_{1,k} - \frac{i}{2}} \right) + \mathcal{O}(g^4)\,. \tag{4.13}$$

We can write the Bethe equation and the momentum constraint in a concise form

$$\left(\frac{u_{j,k} - \frac{i}{2}V_j}{u_{j,k} + \frac{i}{2}V_j}\right)^L \prod_{\substack{j'=1\\ (j',k')\neq(j,k)}}^{N} \prod_{k'=1}^{K_{j'}} \frac{u_{j,k} - u_{j',k'} + \frac{i}{2}M_{j,j'}}{u_{j,k} - u_{j',k'} - \frac{i}{2}M_{j,j'}} = 1\,, \qquad \prod_{j=1}^{N}\prod_{k=1}^{K_j} \frac{u_{j,k} + \frac{i}{2}V_j}{u_{j,k} - \frac{i}{2}V_j} = 1\,. \tag{4.14}$$

Figure 3: Dynkin diagram of $\mathfrak{su}(2|3)$. The signs in the white nodes indicate the sign of the diagonal elements of the Cartan matrix.

Here $N = 4$ is the rank of the symmetry algebra, $V_j = (1, 0, 0, 0)$ are the Dynkin labels of the spin representation and $\mathbf{M}$ is the symmetric Cartan matrix, cf. figure 3.

$$\mathbf{M} = \begin{pmatrix} +2 & -1 & & \\ -1 & +2 & -1 & \\ & -1 & & +1 \\ & & +1 & -2 \end{pmatrix} . \tag{4.15}$$

This is the universal form of Bethe equations for standard quantum spin chains with arbitrary symmetry algebra due to [24 − 26].

4.2 Deformed case

It is not hard to apply the nested Bethe ansatz to the $\mathfrak{u}(2|3)$ R-matrix deformed as in (3.2). Since the graded permutation operator preserves the charge vectors of the incoming and outgoing excitations while only exchanging them, the deformation acts trivially it. We see therefore that only the term in (4.2) involving the identity operator is deformed; consequently, the R-matrix describing the deformed $\mathfrak{u}(2|3)$ spin chain is

$$\tilde{\mathcal{R}}(u)_{12} = \frac{1}{u+i}\left(u\, e^{-i\mathbf{q}_1\times\mathbf{q}_2}\,\mathcal{I}_{12} + i\mathcal{P}_{12}\right) \qquad \tilde{\mathcal{R}}(u)^{kl}_{ij} = \frac{1}{u+i}\left(u\, e^{-iB_{ij}}\,\mathcal{I}^{kl}_{ij} + i\mathcal{P}^{kl}_{ij}\,.\right) \tag{4.16}$$

It is worth emphasizing that, from the standpoint of the spin chain for the undeformed $\mathfrak{u}(2|3)$ algebra, any choice for the matrix $\mathbf{B}$ is allowed. We will take this standpoint in this section and will not commit to a particular form for its matrix elements. In section 5.1 we specialize to the $\mathbf{B}$-matrix following from the discussion in section 2. In some sense this corresponds to removing the central $\mathfrak{u}(1)$ in $\mathfrak{u}(2|3)$ (which is not actually a symmetry of $\mathcal{N} = 4$ SYM) from the set of Cartan generators used for twisting the spin chain.

The simplicity of the deformation (4.16) implies that the modification of the commutation relations following from the Yang-Baxter equations are also minimal:

$$\begin{aligned} \mathcal{A}(u)\,\mathcal{B}^{i_1}(v) &= e^{-B_{i_1 1}}\left[f(v-u)\,\mathcal{B}^{i_1}(v)\,\mathcal{A}(u) - g(v-u)\,\mathcal{B}^{i_1}(u)\,\mathcal{A}(v)\right], && (4.17)\\ \mathcal{D}^{i_1}_{k_1}(u)\,\mathcal{B}^{i_2}(v) &= e^{-B_{k_1 1}}\Big[(-)^{[k_1][j_2]}\, r^{i_1 i_2}_{j_2 j_1}(u-v)\, f(u-v)\,\mathcal{B}^{j_2}(v)\,\mathcal{D}^{j_1}_{k_1}(u) - \\ &\qquad - (-)^{[k_1][i_1]}\, g(u-v)\,\mathcal{B}^{i_1}(v)\,\mathcal{D}^{i_2}_{k_1}(u)\Big], && (4.18)\\ \mathcal{B}^{i_1}(u)\,\mathcal{B}^{i_2}(v) &= r^{i_1 i_2}_{j_2 j_1}(u-v)\,\mathcal{B}^{j_2}(v)\,\mathcal{B}^{j_1}(u)\,. && (4.19) \end{aligned}$$

This in turn leads to inserting phases in the Bethe equations following (5.1).

$$\begin{aligned} 1 &= e^{i(n_1+n_2)B_{21}}\, e^{in_3(B_{23}+B_{31})}\, e^{in_4(B_{24}+B_{41})}\, e^{in_5(B_{25}+B_{51})} \times \\ &\times \left(\frac{u_{1,k}-\frac{i}{2}}{u_{1,k}+\frac{i}{2}}\right)^{L} \prod_{\substack{j=1\\ j\neq k}}^{K_1} \frac{u_{1,k}-u_{1,j}+i}{u_{1,k}-u_{1,j}-i} \prod_{j=1}^{K_2} \frac{u_{1,k}-u_{2,j}-\frac{i}{2}}{u_{1,k}-u_{2,j}+\frac{i}{2}}\,, \end{aligned}$$

$$
\begin{aligned}
1 &= e^{in_1(B_{31}+B_{12})} e^{i(n_2+n_3)B_{32}} e^{in_4(B_{34}+B_{42})} e^{in_5(B_{35}+B_{52})} \times \\
&\times \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}-\frac{i}{2}}{u_{2,k}-u_{1,j}+\frac{i}{2}} \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}+i}{u_{2,k}-u_{2,j}-i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}-\frac{i}{2}}{u_{2,k}-u_{3,j}+\frac{i}{2}}\,, \\
1 &= e^{in_1(B_{41}+B_{13})} e^{in_2(B_{42}+B_{23})} e^{i(n_3+n_4)B_{43}} e^{in_5(B_{45}+B_{53})} \times \\
&\times \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}-\frac{i}{2}}{u_{3,k}-u_{2,j}+\frac{i}{2}} \qquad\qquad \prod_{j=1}^{K_4} \frac{u_{3,k}-u_{4,j}+\frac{i}{2}}{u_{3,k}-u_{4,j}-\frac{i}{2}}\,, \\
1 &= e^{in_1(B_{51}+B_{14})} e^{in_2(B_{52}+B_{24})} e^{in_3(B_{53}+B_{34})} e^{i(n_4+n_5)B_{54}} \times \\
&\times \prod_{j=1}^{K_3} \frac{u_{4,k}-u_{3,j}+\frac{i}{2}}{u_{4,k}-u_{3,j}-\frac{i}{2}} \prod_{\substack{j=1\\ j\neq k}}^{K_4} \frac{u_{4,k}-u_{4,j}-i}{u_{4,k}-u_{4,j}+i}\,.
\end{aligned} \tag{4.20}
$$

Let us now consider the effect of the deformation on the local charge eigenvalues Q_r which appear in the expansion of the transfer matrix eigenvalue $T(u)$ around $u = u_*$

$$
T(u) = \exp i \sum_{r=1}^{\infty} (u - u_*)^{r-1} Q_r\,. \tag{4.21}
$$

Now recall the fact that the eigenvalues of the transfer matrix are given by a sum of five terms, one for each component of the fundamental representation of $\mathfrak{u}(2|3)$. These stem from commuting the $\mathcal{A}$ and diagonal entries of the $\mathcal{D}$ operators past the creation operators in the state (4.9). Furthermore, these five terms are proportional to the eigenvalues of the $\mathcal{A}$ and $\mathcal{D}$ operators corresponding to the vacuum state $|0\rangle$, the latter vanishing for $u = u_*$. Thus, the nontrivial contribution to the eigenvalues of the hamiltonian come from a single term. The deformation is reflected in this term through a multiplicative u-independent phase factor. The expression for the total momentum Q_1 picks up this factor and yields the deformed cyclicity constraint

$$
1 = e^{in_2 B_{21}} e^{in_3 B_{31}} e^{in_4 B_{41}} e^{in_5 B_{51}} \prod_{j=1}^{K_1} \frac{u_{1,j}+\frac{i}{2}}{u_{1,j}-\frac{i}{2}}\,. \tag{4.22}
$$

Beyond that, it has no effect on any of the expressions of the local conserved charges, in particular on the anomalous dimension $\delta D = g^2 Q_2$. Dependence on the deformation parameters comes only through the solutions of the deformed Bethe equations.

We can now translate all the numbers of individual fields n_a to the excitation numbers K_j using (4.12). It is natural and convenient to set $K_0 = L$ with L the length of the spin chain. We therefore introduce the vector

$$
\mathbf{K} = (L|\, K_1, \ldots, K_N)\,, \tag{4.23}
$$

where $N = 4$ is the rank of the symmetry algebra. The elements of the matrix $\mathbf{B}$ then group naturally in a new $(1+N)\times(1+N)$ antisymmetric matrix $\mathbf{A}$ defined by

$$
A_{j,j'} = B_{j,j'} - B_{j,j'+1} - B_{j+1,j'} + B_{j+1,j'+1}\,. \tag{4.24}
$$

Note that $j, j' = 0, \ldots, N$ and we assume $B_{0,*} = B_{*,0} = 0$.

The above twisted Bethe equations (4.20) can now be written conveniently as

$$e^{i(\mathbf{AK})_j}\left(\frac{u_{j,k}-\frac{i}{2}V_j}{u_{j,k}+\frac{i}{2}V_j}\right)^L \prod_{\substack{j'=1\\(j',k')\neq(j,k)}}^{N}\prod_{k'=1}^{K_{j'}} \frac{u_{j,k}-u_{j',k'}+\frac{i}{2}M_{j,j'}}{u_{j,k}-u_{j',k'}-\frac{i}{2}M_{j,j'}}=1 \tag{4.25}$$

and the twisted zero-momentum constraint (4.22) reads

$$e^{i(\mathbf{AK})_0}\prod_{j=1}^{N}\prod_{k=1}^{K_j}\frac{u_{j,k}+\frac{i}{2}V_j}{u_{j,k}-\frac{i}{2}V_j}=1\,. \tag{4.26}$$

Note that we have used a compact vector notation to write the total phases as

$$(\mathbf{AK})_j=A_{j,0}L+\sum_{j'=1}^{N}A_{j,j'}K_{j'}\,,\qquad (\mathbf{AK})_0=\sum_{j=1}^{N}A_{0,j}K_j\,. \tag{4.27}$$

4.3 Deformations of chains in arbitrary representations

We have discussed in detail a class of deformations of spin chains in the fundamental representation of some unitary group focusing on the largest subsector of $\mathcal{N}=4$ SYM with this property — the $\mathfrak{su}(2|3)$ sector — and have seen how the deformation affects the structure of the Bethe equations. However, in the study of the dilatation operator of gauge theories more general spin chains appear, most notably the spin chain based on the self-conjugate doubleton representation of $\mathfrak{psu}(2,2|4)$, describing the dilatation operator of the full $\mathcal{N}=4$ SYM theory [27, 6]. It is natural to ask whether our discussion can be extended to such more general chains and if so how many of the deformation parameters survive.

Following the arguments in section 3 it is not hard to see that all deformation parameters appearing at the level of the R-matrix in the fundamental representation extend without restrictions to more general representations. Indeed, the form of the Yang-Baxter equation is independent of the dimension of the space of states on the sites of the chain. Furthermore, we have shown on general grounds that (3.2) yields solutions of this equation.

Starting from the fundamental representation of $\mathfrak{u}(2,2|4)$, we see that there are at most 28 possible deformation parameters; they form the 8×8 analog of the antisymmetric matrix $\mathbf{C}$ in equation (2.4) which in turn determines the (possibly infinite-dimensional) $\mathbf{B}$-matrix deforming the R-matrix and thus the spin chain hamiltonian. It is worth emphasizing that some of these deformations break Lorentz and conformal symmetry. We will return to this in section 6.

With this input, the diagonalization proceeds following the standard algorithm. First one solves the Yang-Baxter equation for the $\tilde{R}$-matrix acting on the tensor product of the fundamental representation and the representation of interest. From section 3 it follows that this is again a twisted form of the analogous R-matrix in the undeformed theory. Since the sizes of the two Hilbert spaces are now different, the corresponding $\mathbf{B}$-matrix (2.6) is now a rectangular matrix, still determined however by the same 28-parameter $\mathbf{C}$-matrix. Then, the monodromy matrix with the auxiliary Hilbert space in the fundamental representation can be immediately constructed and diagonalized. Due to the Yang-Baxter equation its eigenvectors are also the eigenvectors of the monodromy matrix with the auxiliary Hilbert space in the same representation as the physical sites. Thus, the resulting Bethe equations determine the eigenvalues of the dilatation operator in the representation of interest.

It is certainly possible to follow this procedure and diagonalize the full spin chain for the deformed $\mathcal{N}=4$ SYM theory. Various complications arise from the requirement that resulting eigenstates belong to representations of $\mathfrak{psu}(2,2|4)$; for example, the vacuum which may appear natural from the standpoint of the fundamental representation of the $\mathfrak{u}(2,2|4)$ does not satisfy this requirement. In the following sections we will obtain the deformed Bethe equations by simpler methods. Using those results one may go back and, using the discussion above together with techniques of [28], one may reconstruct the one-loop dilatation operator of the full deformed $\mathcal{N}=4$ SYM theory.

5. Flavor deformations for $\mathcal{N}=4$ SYM

In this section we apply the above findings to the complete spin chain model involving all scalars, fermions, field strengths as well as their covariant derivatives.

5.1 The $\mathfrak{su}(2|3)$ sector

Let us start with the $\mathfrak{su}(2|3)$ sector whose field content is $(\phi_1,\phi_2,\phi_3|\,\chi_1,\chi_2)$. Using the results of section 4.2 it is now straightforward to obtain twist matrices. In terms of the Cartan charges this set of fields is equivalent to the vector $(\Psi_1\Psi_4,\Psi_2\Psi_4,\Psi_3\Psi_4|\,\Psi_4,\Psi_4)$. The matrix of phases $B_{j,j'}=\mathbf{q}_j\times\mathbf{q}_{j'}$ can now be assembled from (2.6)

$$\mathbf{B}=\left(\begin{array}{ccc|cc} 0 & -\gamma_3 & +\gamma_2 & \frac{1}{2}(\gamma_2-\gamma_3) & \frac{1}{2}(\gamma_2-\gamma_3) \\ +\gamma_3 & 0 & -\gamma_1 & \frac{1}{2}(\gamma_3-\gamma_1) & \frac{1}{2}(\gamma_3-\gamma_1) \\ -\gamma_2 & +\gamma_1 & 0 & \frac{1}{2}(\gamma_1-\gamma_2) & \frac{1}{2}(\gamma_1-\gamma_2) \\ \hline \frac{1}{2}(\gamma_3-\gamma_2) & \frac{1}{2}(\gamma_1-\gamma_3) & \frac{1}{2}(\gamma_2-\gamma_1) & 0 & 0 \\ \frac{1}{2}(\gamma_3-\gamma_2) & \frac{1}{2}(\gamma_1-\gamma_3) & \frac{1}{2}(\gamma_2-\gamma_1) & 0 & 0 \end{array}\right). \tag{5.1}$$

Consequently, the phase matrix for the Bethe equations is obtained using (4.24)

$$\mathbf{A}=\left(\begin{array}{c|cccc} 0 & -\gamma_3 & +\gamma_2+\gamma_3 & -\frac{1}{2}\gamma_2-\frac{1}{2}\gamma_3 & 0 \\ \hline +\gamma_3 & 0 & -\gamma_1-\gamma_2-\gamma_3 & +\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2+\gamma_3 & 0 \\ -\gamma_2-\gamma_3 & +\gamma_1+\gamma_2+\gamma_3 & 0 & -\frac{1}{2}\gamma_2-\frac{1}{2}\gamma_3 & 0 \\ +\frac{1}{2}\gamma_2+\frac{1}{2}\gamma_3 & -\frac{1}{2}\gamma_1-\frac{1}{2}\gamma_2-\gamma_3 & +\frac{1}{2}\gamma_2+\frac{1}{2}\gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.2}$$

The phases in the upper-left 3×3 block agree with the phases obtained in [13] in the supersymmetric case $\gamma_1=\gamma_2=\gamma_3$. Also the Bethe equation for the $\mathfrak{su}(2)$ sector (upper-left 2×2 block) agrees with [18].

5.2 The $\mathfrak{so}(6)$ sector

Even though it appears that this sector suffers from the complication we mentioned in section 2 — that the $\mathfrak{su}(4)$ charges are reorganized by the hamiltonian — it is in fact easy

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to see that the apparently problematic term in the R-matrix — the trace operator — is undeformed by (3.2). Consequently, it should not be too complicated to apply the nested Bethe ansatz and find the deformed Bethe equations. Here however we will not follow this route, but rather start from the undeformed Bethe equations and interpret them in terms of the magnon scattering matrix, which is in turn deformed following (3.2).

For reasons which will become clear later we will assume the phase matrix determining the commutation of the the three complex scalars (ϕ_1, ϕ_2, ϕ_3) to be

$$\mathbf{B} = \begin{pmatrix} 0 & +\delta_1 - \delta_3 & -\delta_1 - \delta_3 \\ -\delta_1 + \delta_3 & 0 & +\delta_1 + 2\delta_2 + \delta_3 \\ +\delta_1 + \delta_3 & -\delta_1 - 2\delta_2 - \delta_3 & 0 \end{pmatrix}. \tag{5.3}$$

This is equivalent to the upper left block in (5.1), but with a different parametrization δ_j of the phases γ_j. The new phases are chosen such that the energies are invariant under a shift of any δ_j by 2π. For commuting two conjugate scalars $(\bar{\phi}^1, \bar{\phi}^2, \bar{\phi}^3)$ the same matrix $\mathbf{B}$ applies, while a mixed commutator is determined by $-\mathbf{B}$. This determines all the phases for the $\mathfrak{so}(6)$ spin chain.

Now we have to convert the twist matrix of the fields into a twist matrix for the excitations of the Bethe ansatz. For the rank-three algebra $\mathfrak{so}(6)$, there are three types of excitations, let us denote their creation operators by $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$. As the vacuum we will choose the field ϕ_1. The actions of the three excitations are as follows

$$\begin{aligned} \mathcal{B}_1 &: \phi_2 \mapsto \bar{\phi}^3, \phi_3 \mapsto \bar{\phi}^2 , \\ \mathcal{B}_2 &: \phi_1 \mapsto \phi_2, \bar{\phi}^2 \mapsto \bar{\phi}^1 , \\ \mathcal{B}_3 &: \phi_2 \mapsto \phi_3, \bar{\phi}^3 \mapsto \bar{\phi}^2 . \end{aligned}$$

We would now like to transform the twist matrix to the basis $(\phi_1 | \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ which is, in terms of the charges, equivalent to $(\phi_1 | \bar{\phi}^2\bar{\phi}^3, \bar{\phi}^1\phi_2, \bar{\phi}^2\phi_3)$. We consequently add or subtract the rows and columns according to the composite nature of the excitations $\mathcal{B}_k$ in terms of the ϕ_a. The twist matrix for magnons in the Bethe equations is

$$\mathbf{A} = \left(\begin{array}{c|ccc} 0 & +2\delta_3 & +\delta_1 - \delta_3 & -2\delta_1 \\ \hline -2\delta_3 & 0 & +\delta_1 + 2\delta_2 + 3\delta_3 & -2\delta_1 - 4\delta_2 - 2\delta_3 \\ -\delta_1 + \delta_3 & -\delta_1 - 2\delta_2 - 3\delta_3 & 0 & +3\delta_1 + 2\delta_2 + \delta_3 \\ +2\delta_1 & +2\delta_1 + 4\delta_2 + 2\delta_3 & -3\delta_1 - 2\delta_2 - \delta_3 & 0 \end{array}\right). \tag{5.4}$$

Here the vertical bar separates the components $A_{*,0}$ which couple to the length L from the components $A_{*,j}$ which couple to K_j. The horizontal bar separates the components $A_{0,*}$ for the momentum constraint from the components $A_{j,*}$ which couple to the Bethe equation for $u_{j,k}$. In other words, the deformation of the Bethe equations of [4] is

$$\begin{aligned} 1 &= e^{-2i\delta_3 L} e^{i(+\delta_1+2\delta_2+3\delta_3)K_2} e^{i(-2\delta_1-4\delta_2-2\delta_3)K_3} \times \\ &\quad \times \prod_{\substack{j=1\\ j\neq k}}^{K_1} \frac{u_{1,k} - u_{1,j} + i}{u_{1,k} - u_{1,j} - i} \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} - \frac{i}{2}}{u_{1,k} - u_{2,j} + \frac{i}{2}} , \end{aligned}$$

$$
\begin{aligned}
1 \;&=\; e^{i(-\delta_1+\delta_3)L} e^{i(-\delta_1-2\delta_2-3\delta_3)K_1} e^{i(+3\delta_1+2\delta_2+\delta_3)K_3} \times \\
&\times \left(\frac{u_{2,k}-\frac{i}{2}}{u_{2,k}+\frac{i}{2}}\right)^L \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}-\frac{i}{2}}{u_{2,k}-u_{1,j}+\frac{i}{2}} \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}+i}{u_{2,k}-u_{2,j}-i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}-\frac{i}{2}}{u_{2,k}-u_{3,j}+\frac{i}{2}}\,, \\
1 \;&=\; e^{+2i\delta_1 L} e^{(+2\delta_1+4\delta_2+2\delta_3)K_1} e^{(-3\delta_1-2\delta_2-\delta_3)K_2} \times \\
&\times \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}-\frac{i}{2}}{u_{3,k}-u_{2,j}+\frac{i}{2}} \prod_{\substack{j=1\\ j\neq k}}^{K_3} \frac{u_{3,k}-u_{3,j}+i}{u_{3,k}-u_{3,j}-i}\,,
\end{aligned}
\tag{5.5}
$$

and the cyclicity constraint becomes

$$
1 = e^{+2i\delta_3 K_1} e^{i(+\delta_1-\delta_3)K_2} e^{-2i\delta_1 K_3} \prod_{j=1}^{K_2} \frac{u_{2,j}+\frac{i}{2}}{u_{2,j}-\frac{i}{2}}\,. \tag{5.6}
$$

5.3 Fermions and compatibility with Feynman diagrams

Here we will generalize the results of the previous section to the full $\mathcal{N}=4$ parent theory. In addition to the three scalars ϕ_j transforming canonically under some $\mathfrak{su}(3)$ there are three flavored fermions ψ_j transforming in the same representation and one $\mathfrak{su}(3)$-invariant gluino χ. Let us for the moment generalize the deformation and make it depend not on the charges of the fields, but introduce an independent phase for any set of interacting fields. We deform the couplings of the scalars and the fermions by the phases $\alpha_j, \beta_j, \gamma_j, \alpha'_j, \beta'_j$ (the indices of fields are identified modulo $\mathbb{Z}_3$)[4]

$$
e^{i\alpha'_j}\,\mathrm{Tr}\,\phi_j[\psi_{j+1},\psi_{j-1}]_{\alpha_j}\,, \qquad e^{i\beta'_j}\,\mathrm{Tr}\,\bar\phi^j[\chi,\psi_j]_{\beta_j}\,, \qquad \mathrm{Tr}[\phi_{j-1},\phi_{j+1}]_{\gamma_j}[\bar\phi^{j-1},\bar\phi^{j+1}]_{\gamma_j}\,, \tag{5.7}
$$

with the twisted commutator $[X,Y]_\alpha = e^{+i\alpha/2}XY - e^{-i\alpha/2}YX$. These are the most general renormalizable deformations of the $\mathcal{N}=4$ model using only phases. We now collect the fields in a vector $(\phi_1,\phi_2,\phi_3|\,\psi_1,\psi_2,\psi_3,\chi)$ and determine the twist matrix by combining the Yukawa vertices appropriately

$$
\mathbf{B} = \left(\begin{array}{ccc|cccc}
0 & -\gamma_3 & +\gamma_2 & +\beta_1 & +\alpha_1 & -\alpha_1 & -\beta_1 \\
+\gamma_3 & 0 & -\gamma_1 & -\alpha_2 & +\beta_2 & +\alpha_2 & -\beta_2 \\
-\gamma_2 & +\gamma_1 & 0 & +\alpha_3 & -\alpha_3 & +\beta_3 & -\beta_3 \\
\hline
-\beta_1 & +\alpha_2 & -\alpha_3 & 0 & +\alpha_3 & -\alpha_2 & -\beta_1 \\
-\alpha_1 & -\beta_2 & +\alpha_3 & -\alpha_3 & 0 & +\alpha_1 & -\beta_2 \\
+\alpha_1 & -\alpha_2 & -\beta_3 & +\alpha_2 & -\alpha_1 & 0 & -\beta_3 \\
+\beta_1 & +\beta_2 & +\beta_3 & +\beta_1 & +\beta_2 & +\beta_3 & 0
\end{array}\right). \tag{5.8}
$$

From a field theory point of view, arbitrary values of $\alpha_j, \beta_j, \gamma_j$ are allowed. However, not all values necessarily preserve integrability. To determine the relations among the phases we consider the set of excitations used in the Bethe equations. For the one-loop Bethe equations [6], there are seven types of excitations $\mathcal{B}_j$. These depend on the Cartan

[4] The phases α'_j, β'_j play no role in the diagonal scattering terms investigated below, but for integrability they are necessarily zero, $\alpha'_j = \beta'_j = 0$, due to off-diagonal scattering.

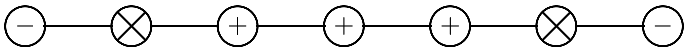

Figure 4: "Beauty" Dynkin diagram of $\mathfrak{su}(2,2|4)$.

matrix which is not uniquely determined for a superalgebra. Here we use the Cartan matrix of $\mathfrak{su}(2,2|4)$ corresponding to the "Beauty" Dynkin diagram [6] in figure 4. Then the excitations $\mathcal{B}_1, \mathcal{B}_7$ merely act on the spacetime part of algebra and there are no deformations. All the other excitations involve flavor degrees of freedom[5]

$$\begin{aligned}
&\mathcal{B}_2 : \bar\phi^3 \mapsto \bar\psi^3,\ \bar\phi^2 \mapsto \bar\psi^2,\ \bar\phi^1 \mapsto \bar\psi^1,\ \psi_3 \mapsto D\phi_3,\ \psi_2 \mapsto D\phi_2,\ \psi_1 \mapsto D\phi_1\,,\\
&\mathcal{B}_3 : \phi_2 \mapsto \bar\phi^3,\ \phi_3 \mapsto \bar\phi^2,\ \chi \mapsto \psi_1,\quad \bar\psi^1 \mapsto \bar\chi\,,\\
&\mathcal{B}_4 : \phi_1 \mapsto \phi_2,\ \bar\phi^2 \mapsto \bar\phi^1,\ \psi_1 \mapsto \psi_2,\ \bar\psi^2 \mapsto \bar\psi^1\,,\\
&\mathcal{B}_5 : \phi_2 \mapsto \phi_3,\ \bar\phi^3 \mapsto \bar\phi^2,\ \psi_2 \mapsto \psi_3,\ \bar\psi^3 \mapsto \bar\psi^2\,,\\
&\mathcal{B}_6 : \phi_3 \mapsto \chi,\quad \bar\phi^2 \mapsto \psi_1,\ \bar\phi^1 \mapsto \psi_2,\ \bar\chi \mapsto D\bar\phi^3,\quad \bar\psi^1 \mapsto D\phi_2,\ \bar\psi^2 \mapsto D\phi_1\,.
\end{aligned} \tag{5.9}$$

When commuting two such excitations we should expect a definite phase in order for integrability to be preserved. However, the excitations can act on various types of fields and depending on the type, there will be a different phase shift when we rely on the most general form (5.8). We therefore have to impose certain constraints on the $\alpha_j, \beta_j, \gamma_j$, namely that the combination of excitations $\mathcal{B}_j\mathcal{B}_j^{-1}$ must be trivial. This should remain true even when choosing two different actions for $\mathcal{B}_j$ and $\mathcal{B}_j^{-1}$ out of the various allowed ones. We can rewrite $\mathcal{B}_j\mathcal{B}_j^{-1}$ as a combination of four fields which should not yield a phase when commuting past anything; we find the following three independent ones

$$\psi_1\phi_2\phi_3\bar\chi \mathrel{\widehat{=}} \psi_2\phi_3\phi_1\bar\chi \mathrel{\widehat{=}} \psi_3\phi_1\phi_2\bar\chi \mathrel{\widehat{=}} 1\,. \tag{5.10}$$

For the purpose of determining the phases, they are as good as any number. They correspond to the following vectors in the space of fields

$$\begin{aligned}
\mathbf{v}_1 &= (0,1,1|\,1,0,0,-1)\,,\\
\mathbf{v}_2 &= (1,0,1|\,0,1,0,-1)\,,\\
\mathbf{v}_3 &= (1,1,0|\,0,0,1,-1)\,.
\end{aligned} \tag{5.11}$$

The consistency condition, i.e. the absence of twists for these combinations, now implies

$$\mathbf{B}\mathbf{v}_1 = \mathbf{B}\mathbf{v}_2 = \mathbf{B}\mathbf{v}_3 = 0\,. \tag{5.12}$$

This leads to the following relations among the phases

$$\begin{aligned}
\alpha_1 &= +\delta_1, & \beta_1 &= +\delta_3, & \gamma_1 &= -\delta_1 - 2\delta_2 - \delta_3,\\
\alpha_2 &= +\delta_1 + \delta_2, & \beta_2 &= -\delta_2 - \delta_3, & \gamma_2 &= -\delta_1 - \delta_3,\\
\alpha_3 &= +\delta_1 + \delta_2 + \delta_3, & \beta_3 &= +\delta_2, & \gamma_3 &= -\delta_1 + \delta_3,
\end{aligned} \tag{5.13}$$

which we have parametrized using the phases δ_j. The twist matrix $\mathbf{B}$ with phases (5.13) therefore leads to an integrable system.

[5]The derivative was included for completeness. For the purposes of the current investigation, it could dropped as it does not carry flavor charges.

We can now transform the twist matrix $\mathbf{B}$ to the form required by the Bethe equations. The new basis is $(\phi_1|\,\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_7)$ and, when expressed in terms of fields, it is equivalent to $(\phi_1|\,1, \phi_1\phi_2\phi_3\bar{\chi}, \bar{\phi}^2\bar{\phi}^3, \bar{\phi}^1\phi_2, \bar{\phi}^2\phi_3, \bar{\phi}^3\chi, 1)$. From (5.8,5.13) we read off the twist matrix for the Bethe equations

$$\mathbf{A} = \left(\begin{array}{c|ccccccc} 0 & 0 & -\delta_3 & +2\delta_3 & +\delta_1-\delta_3 & -2\delta_1 & +\delta_1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +\delta_3 & 0 & 0 & +\delta_3 & -\delta_2-2\delta_3 & +2\delta_2+\delta_3 & -\delta_2 & 0 \\ -2\delta_3 & 0 & -\delta_3 & 0 & +\delta_1+2\delta_2+3\delta_3 & -2\delta_1-4\delta_2-2\delta_3 & +\delta_1+2\delta_2 & 0 \\ -\delta_1+\delta_3 & 0 & +\delta_2+2\delta_3 & -\delta_1-2\delta_2-3\delta_3 & 0 & +3\delta_1+2\delta_2+\delta_3 & -2\delta_1-\delta_2 & 0 \\ +2\delta_1 & 0 & -2\delta_2-\delta_3 & +2\delta_1+4\delta_2+2\delta_3 & -3\delta_1-2\delta_2-\delta_3 & 0 & +\delta_1 & 0 \\ -\delta_1 & 0 & +\delta_2 & -\delta_1-2\delta_2 & +2\delta_1+\delta_2 & -\delta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.14}$$

As can be seen, the middle 3×3 block together with the first row/column agrees with the matrix (5.4). Also, the lower right 4×4 block together with the first row/column agrees, up to the change of variables (5.13), with the matrix (5.2). It therefore follows that the only flavor twist compatible with the Feynman rules is the one discussed in section 2.

The equation (5.14) implies that, for certain choices of twist parameters, nontrivial subalgebras of the superconformal algebra survive in the deformed theory. For $\delta_2 = \delta_3 = 0$, the third row/column disappears. We can then add roots of flavor 2 at infinity $u_2 = \infty$ to the set of Bethe roots without spoiling the equations or changing the energy. This means that not only conformal symmetry $\mathfrak{su}(2,2)$ is preserved, but also supersymmetry, i.e. $\mathcal{N} = 1$ superconformal $\mathfrak{su}(2,2|1)$ symmetry. Similarly, we can set $\delta_1 = \delta_2 = 0$ to preserve (a different) $\mathcal{N} = 1$ supersymmetry. There are three other apparent choices of phases for which some of the symmetry is preserved: $\delta_1 = 2\delta_2 + \delta_3 = 0$, $2\delta_1 + \delta_2 = \delta_2 + 2\delta_3 = 0$ or $\delta_1 + 2\delta_2 = \delta_3 = 0$. In these cases one of the three central rows/columns vanishes and a $\mathfrak{su}(2)$ factor of the internal $\mathfrak{su}(4)$ symmetry survives. Let us summarize the conditions for preserved symmetries

$$\begin{aligned} \mathcal{N} = 1: &\qquad \delta_2 = \delta_3 = 0\,, \\ \mathfrak{su}(2): &\qquad \delta_1 = 2\delta_2 + \delta_3 = 0\,, \\ \mathfrak{su}(2): &\qquad 2\delta_1 + \delta_2 = \delta_2 + 2\delta_3 = 0\,, \\ \mathfrak{su}(2): &\qquad \delta_1 + 2\delta_2 = \delta_3 = 0\,, \\ \mathcal{N} = 1: &\qquad \delta_1 = \delta_2 = 0\,. \end{aligned} \tag{5.15}$$

To improve our understanding of twist matrices, let us see how to obtain $\mathbf{A}$ more directly. We introduce three charge vectors

$$\begin{aligned} \mathbf{q}_{q_1} &= (\ \ 0|\ \ 0,+1,-2,+1,\ \ 0,\ \ 0,\ \ 0), \\ \mathbf{q}_{p} &= (+1|\ \ 0,\ \ 0,+1,-2,+1,\ \ 0,\ \ 0), \\ \mathbf{q}_{q_2} &= (\ \ 0|\ \ 0,\ \ 0,\ \ 0,+1,-2,+1,\ \ 0). \end{aligned} \tag{5.16}$$

These can be used to extract the Dynkin labels $[q_1, p, q_2]$ of a state from the number of excitations $\mathbf{K} = (L|\,K_1, K_2, K_3, K_4, K_5, K_6, K_7)$ by, cf. [6]

$$\mathbf{q}_{q_1}\cdot\mathbf{K} = q_1\,, \qquad \mathbf{q}_p\cdot\mathbf{K} = p\,, \qquad \mathbf{q}_{q_2}\cdot\mathbf{K} = q_2\,. \tag{5.17}$$

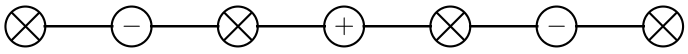

Figure 5: Dynkin diagram of $\mathfrak{su}(2,2|4)$ for the higher-loop Bethe equations.

The matrix $\mathbf{A}$ is then given as follows

$$\mathbf{A} = \delta_1 \left(\mathbf{q}_p \mathbf{q}_{q_2}^{\mathrm{T}} - \mathbf{q}_{q_2} \mathbf{q}_p^{\mathrm{T}}\right) + \delta_2 \left(\mathbf{q}_{q_2} \mathbf{q}_{q_1}^{\mathrm{T}} - \mathbf{q}_{q_1} \mathbf{q}_{q_2}^{\mathrm{T}}\right) + \delta_3 \left(\mathbf{q}_{q_1} \mathbf{q}_p^{\mathrm{T}} - \mathbf{q}_p \mathbf{q}_{q_1}^{\mathrm{T}}\right) . \tag{5.18}$$

5.4 Dualizations

For a superalgebra there are various equivalent forms of the Bethe equations which correspond to various equivalent Dynkin diagrams. The Bethe roots for one state in the various forms are related by dualization. A dualization replaces all Bethe roots of one fermionic type by dualized roots while preserving all the other roots. The new set of roots obeys dualized Bethe equations corresponding to the dualized Dynkin diagram. The procedure of dualization was found in [29, 30] and is described in detail in [31] for the current setup.

We now follow the steps in [31] for a dualization of flavor j and trace the insertions of phases. We see that the phases from the equation for flavor j will be added to the phases in the equations for flavors $j \pm 1$. Afterwards the equation for flavor j is inverted. On the twist matrix this has the following effect on the rows

$$\begin{pmatrix} \vdots \\ A_{j-1,*} \\ A_{j,*} \\ A_{j+1,*} \\ \vdots \end{pmatrix} \to \begin{pmatrix} \vdots \\ A_{j-1,*} + A_{j,*} \\ -A_{j,*} \\ A_{j+1,*} + A_{j,*} \\ \vdots \end{pmatrix} . \tag{5.19}$$

There is however another effect which has to be taken into account: the twist matrix is multiplied to the vector of excitations which changes according to[6]

$$K_j \to K_{j+1} + K_{j-1} - K_j . \tag{5.20}$$

This change is absorbed by the further transformation on columns

$$\begin{pmatrix} \ldots & A_{*,j-1} & A_{*,j} & A_{*,j+1} & \ldots \end{pmatrix} \to \begin{pmatrix} \ldots & A_{*,j-1} + A_{*,j} & -A_{*,j} & A_{*,j+1} + A_{*,j} & \ldots \end{pmatrix} . \tag{5.21}$$

It is the same transformation as (5.19) and thus antisymmetry of $\mathbf{A}$ is preserved.

For example, we can now transform the twist matrix to the Dynkin diagram for the higher-loop Bethe equations used in [21] with $\eta_1 = \eta_2 = +1$, cf. figure 5. This is done most conveniently by preserving the form (5.18) and merely transforming the charge vectors

[6] Here we cannot subtract 1 as in [31] (corresponding to a root at ∞ which we conventionally remove in the undeformed theory), because the leading terms in the polynomial $P(u)$ do not cancel. This is because the breaking of the symmetry of the parent theory. The states in a parent multiplet no longer have a common energy.

(5.16) according to (5.21). The new charge vectors are

$$
\begin{aligned}
\mathbf{q}_{q_1} &= (\ \ 0|-1, \ \ 0,-1,+1, \ \ 0, \ \ 0, \ \ 0), \\
\mathbf{q}_{p} &= (+1| \ \ 0, \ \ 0,+1,-2,+1, \ \ 0, \ \ 0), \\
\mathbf{q}_{q_2} &= (\ \ 0| \ \ 0, \ \ 0, \ \ 0,+1,-1, \ \ 0,-1)
\end{aligned}
\tag{5.22}
$$

and the twist matrix is

$$
\mathbf{A} = \left(\begin{array}{c|ccccccc}
0 & +\delta_3 & 0 & +\delta_3 & +\delta_1-\delta_3 & -\delta_1 & 0 & -\delta_1 \\
\hline
-\delta_3 & 0 & 0 & -\delta_3 & +\delta_2+2\delta_3 & -\delta_2-\delta_3 & 0 & -\delta_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\delta_3 & +\delta_3 & 0 & 0 & +\delta_1+\delta_2+\delta_3 & -\delta_1-\delta_2-\delta_3 & 0 & -\delta_1-\delta_2 \\
-\delta_1+\delta_3 & -\delta_2-2\delta_3 & 0 & -\delta_1-\delta_2-\delta_3 & 0 & +\delta_1+\delta_2+\delta_3 & 0 & +2\delta_1+\delta_2 \\
+\delta_1 & +\delta_2+\delta_3 & 0 & +\delta_1+\delta_2+\delta_3 & -\delta_1-\delta_2-\delta_3 & 0 & 0 & -\delta_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+\delta_1 & +\delta_2 & 0 & +\delta_1+\delta_2 & -2\delta_1-\delta_2 & \delta_1 & 0 & 0
\end{array}\right).
\tag{5.23}
$$

In this form one row/column vanishes when we set

$$
\begin{aligned}
\mathcal{N}=1: &\qquad \delta_2=\delta_3=0\,, \\
\mathcal{N}=1: &\qquad \delta_1+\delta_2=\delta_3=0\,, \\
\mathfrak{su}(2): &\qquad 2\delta_1+\delta_2=\delta_2+2\delta_3=0\,, \\
\mathcal{N}=1: &\qquad \delta_1=\delta_2+\delta_3=0=0\,, \\
\mathcal{N}=1: &\qquad \delta_1=\delta_2=0\,.
\end{aligned}
\tag{5.24}
$$

Note that as compared to (5.15) two of the conditions have been replaced by different ones which now preserve $\mathcal{N}=1$ supersymmetry instead of $\mathfrak{su}(2)$. As both forms of the Bethe equations are equivalent, all the relations between the angles in (5.15,5.24) must lead to symmetries. Some of the symmetries are however hidden in one form and manifest in the other.

5.5 Higher loops

It has turned out that the planar dilatation operator of $\mathcal{N}=4$ SYM is not only integrable at one loop, but also at higher loops [5], i.e. higher order in the coupling constant

$$
g^2 = \frac{\lambda}{8\pi^2}\,.
\tag{5.25}
$$

In [21] Bethe equations for this long-range chain have been proposed. These are a generalization of the Bethe equations for the $\mathfrak{su}(2)$ sector at higher loops [32, 33] which are somewhat similar to the ones of the Inozemtsev spin chain [34, 35]. Let us briefly summarize the equations and refer the reader to [21] for all details. The most convenient way of parametrizing the Bethe equations is to treat the rapidities $u_{j,k}$ as composite quantities [36] related to the more fundamental variables $x_{j,k}$ by

$$
u_{j,k} = u(x_{j,k})\,, \qquad u(x) = x + \frac{g^2}{2x}\,.
\tag{5.26}
$$

Similarly, we define the derived quantity $x^{\pm}_{j,k}$ as

$$x^{\pm}_{j,k} = x(u_{j,k} \pm \tfrac{i}{2})\,, \qquad x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1 - \frac{2g^2}{u^2}}\,. \tag{5.27}$$

The higher-loop Bethe equations and momentum constraint proposed in [21] read

$$U_0 = 1\,, \qquad U_j(x_{j,k}) \prod_{\substack{j'=1 \\ (j',k')\neq(j,k)}}^{7} \prod_{k'=1}^{K_{j'}} \frac{u_{j,k} - u_{j',k'} + \frac{i}{2}M_{j,j'}}{u_{j,k} - u_{j',k'} - \frac{i}{2}M_{j,j'}} = 1\,, \tag{5.28}$$

with $M_{j,j'}$ the Cartan matrix specified by figure 5. The deformation of the (untwisted) one-loop equations (4.25,4.26) is contained in the terms U_j which read

$$U_0 = \prod_{k=1}^{K_4} \frac{x^+_{4,k}}{x^-_{4,k}}\,, \qquad U_1(x) = U_3^{-1}(x) = U_5^{-1}(x) = U_7(x) = \prod_{k=1}^{K_4} S_{\mathrm{aux}}(x_{4,k}, x) \tag{5.29}$$

as well as

$$U_4(x) = U_{\mathrm{s}}(x) \left(\frac{x^-}{x^+}\right)^L \prod_{k=1}^{K_1} S^{-1}_{\mathrm{aux}}(x, x_{1,k}) \prod_{k=1}^{K_3} S_{\mathrm{aux}}(x, x_{3,k}) \prod_{k=1}^{K_5} S_{\mathrm{aux}}(x, x_{5,k}) \prod_{k=1}^{K_7} S^{-1}_{\mathrm{aux}}(x, x_{7,k}). \tag{5.30}$$

The auxiliary scattering term $S_{\mathrm{aux}}(x_1, x_2)$ is defined as

$$S_{\mathrm{aux}}(x_1, x_2) = \frac{1 - g^2/2x_1^+ x_2}{1 - g^2/2x_1^- x_2}\,. \tag{5.31}$$

The dressing factor $U_{\mathrm{s}}(x)$ is some function of the $x_{4,k}$ which can be used to deform the model. For gauge theory it is trivial, $U_{\mathrm{s}}(x) = 1$. The anomalous dimension of a state is given by

$$\delta D = g^2 \sum_{k=1}^{K_4} \left(\frac{i}{x^+_{4,k}} - \frac{i}{x^-_{4,k}}\right)\,. \tag{5.32}$$

A complication of the higher-loop model is that the length of the chain is not preserved by the hamiltonian, the spin chain becomes *dynamic* at higher loops [20]. It was argued that the proposed Bethe equations have a symmetry which enables one to interpret a set of Bethe roots as a spin chain state with flexible length. This *dynamic transformation* is a prescription of how to change a set of Bethe roots for length L into a set of Bethe roots for length $L + 1$. It replaces a root of type 3 by a root of type 1

$$x_3 \to x_1 = \frac{g^2}{2x_3} \tag{5.33}$$

so that the excitation numbers change according to

$$K_3 \to K_3 - 1\,, \qquad K_1 \to K_1 + 1\,, \qquad L \to L + 1\,, \qquad B \to B - 1\,. \tag{5.34}$$

Here, B is the hypercharge of the state. The transformation is a symmetry of the Bethe equations due to the identities

$$u(x_3) = u\left(\frac{g^2}{2x_3}\right), \qquad \frac{x_{4,k}^+}{x_{4,k}^-} S_{\mathrm{aux}}(x_{4,k}, x_3) S_{\mathrm{aux}}\left(x_{4,k}, \frac{g^2}{2x_3}\right) = \frac{u_{4,k} - u_3 + \frac{i}{2}}{u_{4,k} - u_3 - \frac{i}{2}}\,. \tag{5.35}$$

This proves the invariance of the Bethe equation for $x_{4,k}$ and leads to the identity

$$U_3(x_3) \prod_{k=1}^{K_4} \frac{u_3 - u_{4,k} - \frac{i}{2}}{u_3 - u_{4,k} + \frac{i}{2}} = U_0\, U_1 \left(\frac{g^2}{2x_3}\right) \tag{5.36}$$

which relates the Bethe equations of x_3 and $x_1 = g^2/x_3$. Note that $M_{3,4} = -1$ and $M_{1,4} = 0$. An analogous transformation replaces a root of type 5 by a root of type 7

$$K_5 \to K_5 - 1\,, \qquad K_7 \to K_7 + 1\,, \qquad L \to L + 1\,, \qquad B \to B + 1\,. \tag{5.37}$$

We would now like to introduce the twist into the higher-loop model. The twisting of the spin chain is a discrete procedure: there is a fixed phase for an interchange of a pair of fields. Moreover, the phase depends only on the conserved charges of the fields. This implies that the twisting is independent of the loop order. In particular it can also be inferred from the position space Bethe ansatz in [37, 21]. Let us therefore twist the higher-loop Bethe equations in the most obvious way, we multiply the Bethe equations and momentum constraint (5.28) by the appropriate phases as in (4.25,4.26)

$$e^{i(\mathbf{AK})_0}\, U_0 = 1\,, \qquad e^{i(\mathbf{AK})_j}\, U_j(x_{j,k}) \prod_{\substack{j'=1 \\ (j',k')\neq(j,k)}}^{7} \prod_{k'=1}^{K_{j'}} \frac{u_{j,k} - u_{j',k'} + \frac{i}{2} M_{j,j'}}{u_{j,k} - u_{j',k'} - \frac{i}{2} M_{j,j'}} = 1\,. \tag{5.38}$$

This deformation is consistent with the twist [18] of the higher-loop Bethe equations in the $\mathfrak{su}(2)$ sector [38].

Does this interfere with the dynamic transformation? Let us first consider the Bethe equations for those roots which are not transformed. According to (5.34) we increase each of $L, K_1, -K_3$ by one unit. As it turns out, the sum of the first and second column in (5.23) equals the fourth column. Therefore these Bethe equations are not modified by the transformation. We need to confirm that also the Bethe equation for the transformed root does not change. The total twist in Bethe equations for roots of type 1 and type 3 is different. However, even the untwisted equations are not immediately the same, but only after making use of the momentum constraint. Now the momentum constraint is twisted as well, cf. (4.26), and it accounts for the difference of phases. In (5.23) we can see that the first row (which enters the momentum constraint) plus the third row equals the fourth row. In fact, this is obvious due to the antisymmetry of $\mathbf{A}$ and the above observations. A similar argument holds for the other dynamic transformation (5.37). Twisting of the proposed higher-loop Bethe equations is therefore consistent.

The deeper reason why the twist is possible is that it only depends on the Dynkin labels $[q_1, p, q_2]$ of the state and not on on the length L and the hypercharge B. The Dynkin labels are physical and they are invariant under the dynamic transformation. Put differently, the combination which enters the Bethe equations, $\mathbf{AK}$, depends on $[q_1, p, q_2]$ only, but not on

L, B. Curiously, we did not explicitly use this constraint when deriving (5.23). Instead we demanded compatibility of Feynman rules with integrability. This however implicitly leads to the independence of L and B because the diagrams do not conserve these quantities.

6. Deformations involving spacetime

In section 4.3 we have investigated the most general twisting on a standard integrable spin chain. We have seen that it is specified by a generic antisymmetric matrix $\mathbf{A}$ with 28 free parameters. In section 5.3 we have then seen that not every such twisted standard integrable spin chain can be realized at one loop by a twist of $\mathcal{N}=4$ SYM. Conversely, not every twist of $\mathcal{N}=4$ SYM has a one loop dilatation operator which is integrable. Here we would like to investigate the set of all integrable twists of $\mathcal{N}=4$ SYM.

6.1 Bethe ansatz

The starting point will be the independence of $\mathbf{AK}$ of the length L and the hypercharge B. In section section 5.3 this was a consequence of the requirement of compatibility of Feynman diagrammatics and integrability. As we have seen in section 5.5, this independence is necessary for consistency of the higher-loop Bethe equations. We will therefore require it in the following.

To construct a matrix $\mathbf{A}$, we shall use a form similar to (5.18)

$$\mathbf{A}=\sum_{j,j'}\alpha_{j,j'}\left(\mathbf{q}_j\mathbf{q}_{j'}^{\mathrm{T}}-\mathbf{q}_{j'}\mathbf{q}_j^{\mathrm{T}}\right), \tag{6.1}$$

where the $\mathbf{q}_j$ are the allowed charge vectors. We have used the charge vectors (6.2)

$$\begin{aligned}
\mathbf{q}_{q_1} &= (\ \ 0|-1,\ \ 0,-1,+1,\ \ 0,\ \ 0,\ \ 0),\\
\mathbf{q}_{p} &= (+1|\ \ 0,\ \ 0,+1,-2,+1,\ \ 0,\ \ 0),\\
\mathbf{q}_{q_2} &= (\ \ 0|\ \ 0,\ \ 0,\ \ 0,+1,-1,\ \ 0,-1)
\end{aligned} \tag{6.2}$$

dual to the Dynkin labels $[q_1,p,q_2]$ of $\mathfrak{su}(4)$. Similarly, we could use the charge vectors

$$\begin{aligned}
\mathbf{q}_{s_1} &= (\ \ 0|-2,+1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0),\\
\mathbf{q}_{r_0} &= (-1|+1,-1,\ \ 0,\ \ 0,\ \ 0,-1,+1),\\
\mathbf{q}_{s_2} &= (\ \ 0|\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,+1,-2),
\end{aligned} \tag{6.3}$$

which are dual to the Dynkin labels $[s_1,r,s_2]$ of the conformal algebra $\mathfrak{su}(2,2)$. The labels s_1,s_2 are twice the Lorentz spins and $r=-D-\frac{1}{2}s_1-\frac{1}{2}s_2$ with D the scaling dimension. To complete the basis of the eight-dimensional space we introduce

$$\begin{aligned}
\mathbf{q}_{L} &= (+1|\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0),\\
\mathbf{q}_{B} &= \left(\ \ 0|-\frac{1}{2},\ \ 0,+\frac{1}{2},\ \ 0,-\frac{1}{2},\ \ 0,+\frac{1}{2}\right),
\end{aligned} \tag{6.4}$$

which are dual to L and B. Clearly, the latter two are not suitable for the construction of $\mathbf{A}$ in (6.1).[7]

[7] They can be used for the one-loop Bethe equations but these equations do not correspond to a deformation of $\mathcal{N}=4$ SYM. The higher-loop Bethe equations are incompatible with $\mathbf{q}_L$ and $\mathbf{q}_B$.

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It is questionable whether we should use $\mathbf{q}_{r_0}$ or not. All the other labels q_1, p, q_2, s_1, s_2 are positive integers. Conversely, $r = -D_0 - \frac{1}{2}s_1 - \frac{1}{2}s_2 - \delta D$ is irrational in almost all cases due to the anomalous dimension δD. However, in the way we have twisted the equations, $\mathbf{q}_{r_0}$ does not actually couple to r, but to its classical limit

$$\mathbf{q}_{r_0} \cdot \mathbf{K} = r_0 = -D_0 - \frac{1}{2}s_1 - \frac{1}{2}s_2 \,. \tag{6.5}$$

We could also allow for twisting using the exact label r or, equivalently, using the anomalous dimension δD. From a spin chain point of view, this may be possible and will yield deformations (see [21] for notation)

$$\exp\left(ig^2 Q_2(g)\right) \qquad \text{or} \qquad \exp\left(ig^2 K_j q_2(x_{4,k})\right) \,, \tag{6.6}$$

which are somewhat similar to the deformation discussed in [38]. Theoretically, one might also consider combinations of the various higher charges $q_s(x_j)$ of the original model. All of these generalizations are qualitatively different from the ones discussed in this paper because they explicitly refer to the positions $x_{j,k}$ of the Bethe roots. Here we will not discuss them further.

6.2 A non-commutative gauge theory

In the previous subsection we suggested that it is possible to deform the spin chain of the $\mathcal{N} = 4$ theory such that its integrability is preserved while the lagrangian of the deformed field theory is not Lorentz invariant. These deformations appear to fall in the class of deformations introduced in section 3 and it is perhaps interesting to better understand the structure of the deformed field theory leading to them.

It is certainly possible to extend the construction [14] of the supergravity duals of deformations of the type discussed in section 2 to include breaking of Lorentz and conformal invariance. Indeed, all one has to do is to identify the three commuting isometries of the AdS space and include them in the T-duality–shift–T-duality sequence of transformations. It is known from [39, 40] that such transformations performed along the isometries corresponding to translations along the boundary of the AdS space in the Poincaré patch correspond to the Moyal deformation of the boundary theory. Our situation is different, however, since we are interested in using the AdS isometries corresponding to the Cartan generators of the four-dimensional superconformal group rather than the shift symmetries manifest on the Poincaré patch. Nevertheless, we expect to find some kind of noncommutative field theory.

Before discussing the deformation of the lagrangian it is important to decide the space-time on which the theory is defined. In the case of the $\mathcal{N} = 4$ theory, conformal transformations map the theory defined on $\mathbb{R}^4$ and on $S^3 \times \mathbb{R}$ into each other. The deformation however breaks conformal symmetry and thus the deformation of the theory on the plane is in principle physically different from the deformation of the theory defined on $S^3 \times \mathbb{R}$ and it is not *a priori* clear which one corresponds to the deformed spin chain introduced in the previous subsection.

Starting from the properties of the potential gravity dual of the deformed theory and borrowing from the experience with the Lorentz-preserving deformations it is fair to guess that, in its most general form, the deformation we are looking for replaces the ordinary product of fields by

$$XY \;\mapsto\; X * Y = e^{iC_{ab}h^a_X h^b_Y/2}\, X\, Y\,, \tag{6.7}$$

where $h^a_{X,Y}$ are the Cartan generators of $\mathfrak{psu}(2,2|4)$ acting on either X or Y. The fact that the generators of the Cartan subalgebra commute among themselves and that they act following the Leibniz rule makes this $*$-product somewhat similar with the Moyal product and thus it is associative.

In writing this expression we assumed that, from a string theory perspective, we are allowed to perform T-duality transformations along any isometry, in particular that we can T-dualize the time direction. This is usually problematic, since it yields the wrong sign for the kinetic terms of certain RR fields [41, 42] and can be used only as a solution-generating technique. It is therefore questionable whether we should use it at all. This is similar to whether we should use the charge vector $\mathbf{q}_{r_0}$ for twisting the gauge theory spin chain.

In the case at hand a carefully-chosen deformation parameter — such that both the original and shifted coordinates are time-like — may render harmless the problem of the positivity of the RR kinetic terms, so we will cautiously proceed along this line. The gauge theory symmetry generator corresponding to global time translations is a combination of the generator of scale transformations and a special conformal generator [43]. Since the dilatation operator receives quantum corrections, it is not clear whether the corresponding h appearing in (6.7) should be the quantum-corrected operator or only the classical part. However, since the fundamental fields of the theory are not gauge invariant, their anomalous dimensions exhibit gauge-fixing dependence ambiguities. This suggests that, with the appropriate gauge-fixing, we may choose the tree-level generator of scale transformations to appear in (6.7). This is analogous to the observation in the previous section that $\mathbf{q}_{r_0}$ couples to the classical limit of the Dynkin label r.

It is quite problematic to analyze in perturbation theory the consequences of the deformation (6.7) of a field theory on $\mathbb{R}^4$. Perhaps the main obstacle is that the Cartan generators depend explicitly on coordinates, $h \sim x\partial_y - y\partial_x$ and thus the deformed lagrangian is position dependent. Consequently, the momentum space calculations — which are useful for ordinary noncommutative theories — become inefficient for this kind of deformation.

On $S^3 \times \mathbb{R}$ the situation is somewhat better because the lagrangian is not anymore position-dependent. Indeed, the Cartan generators correspond to the two commuting isometries of $S^3 \simeq \mathrm{SO}(4)/\mathrm{SO}(3)$ and translations along $\mathbb{R}$. In this form they do not exhibit explicit position dependence and thus on $S^3 \times \mathbb{R}$ the deformation (6.7) closely resembles the Moyal deformation. Explicit calculations are relatively difficult however and perhaps the most efficient avenue involves dimensionally reducing the theory to a quantum mechanical model with infinitely many fields by expanding the four-dimensional fields in spherical harmonics on S^3. Each such mode is clearly an eigenvector of the two Cartan generators of SO(4) and, up to the cautionary words on time-like T-duality and the identification of the generator dual to global time translations in AdS space, of the third generator as well.

Thus, from the standpoint of this matrix quantum mechanics model, the deformation (6.7) appears identical (up to phases involving the classical dimension of fields) with the one involving the Cartan generators of the internal symmetry group. This is, of course, not surprising since from the standpoint of the $(0+1)$-dimensional quantum mechanical model, Lorentz and internal symmetry transformations are on equal footing.

We will not write out explicitly the expression of the deformed field theory lagrangian. Clearly, it is quite complicated to directly compute the dilatation operator and thus derive the spin chain hamiltonian from first principles. It is however straight-forward to use the deformed R-matrix to derive the planar dilatation operator at one loop and it is probably possible to generalize it to include non-planar interactions and thus generalize the calculation [44] of correlation functions. Nevertheless, for the planar theory it is not necessary to have the hamiltonian explicitly, but use the Bethe ansatz outlined in (6.1) to determine the energy eigenvalues, not only at one-loop, but probably at higher loops as well. The benefit of this approach is that the hamiltonian neither has to be computed nor evaluated. Both alternative steps would be extremely laborious when it comes to higher loops and long local operators, see [7].

It is also not completely clear what the energy eigenvalues from the Bethe ansatz correspond to in the noncommutative field theory. They should be the generalization of anomalous dimensions of local operators for a noncommutative theory. However, there are at least two ways of constructing local composite operators in such a model. One could either stick to the original definition, e.g.

$$\mathcal{O} = \mathrm{Tr}\, \phi_{k_1}\phi_{k_2}\phi_{k_3}\ldots\phi_{k_L} + \cdots \tag{6.8}$$

or use the $*$-product to concatenate the fields, e.g.

$$\mathcal{O} = \mathrm{Tr}\, \phi_{k_1} * \phi_{k_2} * \phi_{k_3} * \ldots * \phi_{k_L} + \cdots\,. \tag{6.9}$$

The two pictures differ slightly although the energy eigenvalues should agree in the end (see also [13]): In the first, the action of the spin chain hamiltonian is deformed uniformly. For the second, it is well-known that the spin chain hamiltonian (i.e. the planar interactions) is the same as for the commutative model. This is at least almost true, but the interaction that wraps the trace in (6.9) is deformed, so here the hamiltonian has a defect localized at the end of the trace. Similarly, the trace in (6.8) is manifestly cyclic, whereas the one in (6.9) is not: the fields can be permuted cyclically, but not in any trivial way.

One can also cast the Bethe ansatz in both pictures. A uniformly deformed hamiltonian would give rise to twists whenever two excitations cross their path. A term in the Bethe equation would thus be written as

$$\prod_{k'=1}^{K_{j'}} \left(e^{iA_{j,j'}} \frac{u_{j,k} - u_{j',k'} + \frac{i}{2}M_{j,j'}}{u_{j,k} - u_{j',k'} - \frac{i}{2}M_{j,j'}} \right). \tag{6.10}$$

When the deformation of the hamiltonian is localized, the phase shifts for crossing excitations are not modified. However, we have to consider the effect of excitations stepping across the defect. In the Bethe equations this would manifest as the term

$$e^{iA_{j,j'}K_{j'}}\prod_{k'=1}^{K_{j'}}\frac{u_{j,k}-u_{j',k'}+\frac{i}{2}M_{j,j'}}{u_{j,k}-u_{j',k'}-\frac{i}{2}M_{j,j'}}\,. \tag{6.11}$$

Clearly both pictures are equivalent and lead to the same final results.

7. Discussion

In this article we have discussed a general method of twisting integrable spin chains which is dual to the twist described in [14] on the supergravity side. We have started out by describing the deformation of a generic integrable quantum spin chain and its R-matrix. It is based on modifying the phase of the R-matrix depending on the Cartan charges of the involved sites. We have shown that the deformed R-matrix still satisfies the Yang-Baxter equation and therefore our twist is an integrable deformation. From the R-matrix for a spin chain with $\mathfrak{u}(2|3)$ symmetry we have derived a set of twisted Bethe equations which generalizes in a straight-forward fashion to arbitrary symmetry algebras.

The most general integrable twist is, however, not compatible with the Feynman rules of twisted $\mathcal{N}=4$ SYM. Conversely, the most general deformation of the interactions of $\mathcal{N}=4$ SYM breaks integrability. We have derived the intersection of these two requirements and obtained the deformation of the Bethe equations for the complete $\mathcal{N}=4$ spin chain model at one loop. Excitingly, the deformation can be applied directly to the higher-loop Bethe ansatz proposed in [21] without corrupting its remarkable properties. It is interesting to see that this depends crucially on the constraints from compatibility with Feynman rules; it may be taken as an indication that the higher-loop Bethe ansatz indeed respects the Feynman rules of the underlying field theory.

Finally, we have generalized the twist to include deformations not only of flavor symmetry, but also of spacetime symmetries. The resulting model is a sort of noncommutative field theory. On $\mathbb{R}^4$ it differs from the usual noncommutative theories in that the $*$-product employs the Cartan generators of the Lorentz group rather than the momentum generators. Alternatively, when using $S^3\times\mathbb{R}$ as undeformed spacetime, it becomes an ordinary noncommutative theory with noncommutativity involving two of the angles on S^3. Especially here, but also for the theory with twists restricted to flavor, the Bethe equations are a valuable tool: They allow to obtain planar anomalous dimensions without having to go through very laborious field theory computations, which would be particularly cumbersome for the noncommutative theory.

It would be interesting to compare our results to the twisted analog of the integrable structure of the classical superstring sigma model on $AdS_5\times S^5$ [9]. For that one would have to repeat the construction of the spectral curve and the finite gap method of [45, 46] for the twisted model proposed in [14]. The resulting integral equations could then be compared to the thermodynamic limit of the algebraic equations for the string chain [38, 36] proposed in [21].

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